

ON SOLUTION-FREE SETS OF INTEGERS

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ABSTRACT. Given a linear equation \mathcal{L} , a set $A \subseteq [n]$ is \mathcal{L} -free if A does not contain any ‘non-trivial’ solutions to \mathcal{L} . In this paper we consider the following three general questions:

- (i) What is the size of the largest \mathcal{L} -free subset of $[n]$?
- (ii) How many \mathcal{L} -free subsets of $[n]$ are there?
- (iii) How many maximal \mathcal{L} -free subsets of $[n]$ are there?

We completely resolve (i) in the case when \mathcal{L} is the equation $px + qy = z$ for fixed $p, q \in \mathbb{N}$ where $p \geq 2$. Further, up to a multiplicative constant, we answer (ii) for a wide class of such equations \mathcal{L} , thereby refining a special case of a result of Green [18]. We also give various bounds on the number of maximal \mathcal{L} -free subsets of $[n]$ for three-variable homogeneous linear equations \mathcal{L} . For this, we make use of container and removal lemmas of Green [18].

1. INTRODUCTION

Let $[n] := \{1, \dots, n\}$ and consider a fixed linear equation \mathcal{L} of the form

$$(1) \quad a_1x_1 + \dots + a_kx_k = b$$

where $a_1, \dots, a_k, b \in \mathbb{Z}$. If $b = 0$ we say that \mathcal{L} is *homogeneous*. If

$$\sum_{i \in [k]} a_i = b = 0$$

then we say that \mathcal{L} is *translation-invariant*. Let \mathcal{L} be translation-invariant. Then notice that (x, \dots, x) is a ‘trivial’ solution of (1) for any x . More generally, a solution (x_1, \dots, x_k) to \mathcal{L} is said to be *trivial* if there exists a partition P_1, \dots, P_ℓ of $[k]$ so that:

- (i) $x_i = x_j$ for every i, j in the same partition class P_r ;
- (ii) For each $r \in [\ell]$, $\sum_{i \in P_r} a_i = 0$.

A set $A \subseteq [n]$ is \mathcal{L} -free if A does not contain any non-trivial solutions to \mathcal{L} . If the equation \mathcal{L} is clear from the context, then we simply say A is *solution-free*.

The notion of an \mathcal{L} -free set encapsulates many fundamental topics in combinatorial number theory. Indeed, in the case when \mathcal{L} is $x_1 + x_2 = x_3$ we call an \mathcal{L} -free set a *sum-free set*. This is a notion that dates back to 1916 when Schur [34] proved that, if n is sufficiently large, any r -colouring of $[n]$ yields a monochromatic triple x, y, z such that $x + y = z$. *Sidon sets* (when \mathcal{L} is $x_1 + x_2 = x_3 + x_4$) have also been extensively studied. For example, a classical result of Erdős and Turán [16] asserts that the largest Sidon set in $[n]$ has size $(1 + o(1))\sqrt{n}$. In the case when \mathcal{L} is $x_1 + x_2 = 2x_3$ an \mathcal{L} -free set is simply a *progression-free set*. Roth’s theorem [27] states that the largest progression-free subset of $[n]$ has size $o(n)$. In [28, 29], Ruzsa instigated the study of solution-free sets for general linear equations.

In this paper we prove a number of results concerning \mathcal{L} -free subsets of $[n]$ where \mathcal{L} is a homogeneous linear equation in *three variables*. In particular, our work is motivated by the following general questions:

- (i) What is the size of the largest \mathcal{L} -free subset of $[n]$?
- (ii) How many \mathcal{L} -free subsets of $[n]$ are there?

(iii) How many *maximal* \mathcal{L} -free subsets of $[n]$ are there?

We make progress on all three of these questions. For each question we use tools from graph theory; for (i) and (ii) our methods are somewhat elementary. For (iii) our method is more involved and utilises container and removal lemmas of Green [18].

1.1. The size of the largest solution-free set. As highlighted above, a central question in the study of \mathcal{L} -free sets is to establish the size $\mu_{\mathcal{L}}(n)$ of the largest \mathcal{L} -free subset of $[n]$. It is not difficult to see that the largest sum-free subset of $[n]$ has size $\lceil n/2 \rceil$, and this bound is attained by the set of odd numbers in $[n]$ and by the interval $[\lceil n/2 \rceil + 1, n]$.

When \mathcal{L} is $x_1 + x_2 = 2x_3$, $\mu_{\mathcal{L}}(n) = o(n)$ by Roth's theorem. In fact, very recently Bloom [9] proved that there is a constant C such that every set $A \subseteq [n]$ with $|A| \geq Cn(\log \log n)^4 / \log n$ contains a three-term arithmetic progression. On the other hand, Behrend [7] showed that there is a constant $c > 0$ so that $\mu_{\mathcal{L}}(n) \geq n \exp(-c\sqrt{\log n})$. See [15, 19] for the best known lower bound on $\mu_{\mathcal{L}}(n)$ in this case.

More generally, it is known that $\mu_{\mathcal{L}}(n) = o(n)$ if \mathcal{L} is translation-invariant and $\mu_{\mathcal{L}}(n) = \Omega(n)$ otherwise (see [28]). For other (exact) bounds on $\mu_{\mathcal{L}}(n)$ for various linear equations \mathcal{L} see, for example, [28, 29, 6, 14, 21].

In this paper we mainly focus on \mathcal{L} -free subsets of $[n]$ for linear equations \mathcal{L} of the form $px + qy = z$ where $p \geq 2$ and $q \geq 1$ are fixed integers. Notice that for such a linear equation \mathcal{L} , the interval $[\lceil n/(p+q) \rceil + 1, n]$ is an \mathcal{L} -free set. Our first result implies that this is the largest such \mathcal{L} -free subset of $[n]$. Let $\min(S)$ denote the smallest element in a finite set $S \subseteq \mathbb{N}$.

Theorem 1. *Let \mathcal{L} denote the equation $px + qy = z$ where $p \geq q$ and $p \geq 2, p, q \in \mathbb{N}$. Let S be an \mathcal{L} -free subset of $[n]$, and let $\min(S) = \lfloor \frac{n}{p+q} \rfloor - t$ where t is a non-negative integer.*

(i) *If $0 \leq t < (\frac{p+q-1}{p+q+p/q}) \lfloor \frac{n}{p+q} \rfloor$ then $|S| \leq \lceil \frac{(p+q-1)n}{p+q} \rceil - \lfloor \frac{p}{q}t \rfloor$.*

(ii) *If $t \geq (\frac{p+q-1}{p+q+p/q}) \lfloor \frac{n}{p+q} \rfloor$ then $|S| \leq \frac{(q^2+1)n}{q^2+q+1}$ provided that*

$$n \geq \max \left\{ \frac{3(q^2 + q + 1)(q^3 + p(q^2 + q + 1))}{q^2 + 1}, \frac{5(q^2 + q + 1)(q^5 + p(q^4 + q^3 + q^2 + q + 1))}{q^4 + (p-1)q^3 + q^2 + 1} \right\}.$$

In both cases of Theorem 1 we observe that $|S| \leq n - \lfloor \frac{n}{p+q} \rfloor$, hence the following corollary holds.

Corollary 2. *Let \mathcal{L} denote the equation $px + qy = z$ where $p \geq q$ and $p \geq 2, p, q \in \mathbb{N}$. If n is sufficiently large then $\mu_{\mathcal{L}}(n) = n - \lfloor \frac{n}{p+q} \rfloor$.*

Roughly, Theorem 1 implies that every \mathcal{L} -free subset of $[n]$ is ‘interval like’ or ‘small’. In the case of sum-free subsets (i.e. when $p = q = 1$), a result of Deshouillers, Freiman, Sós and Temkin [13] provides very precise structural information on the sum-free subsets of $[n]$. Loosely speaking, they showed that a sum-free subset of $[n]$ is ‘interval like’, ‘small’ or consists entirely of odd numbers.

In the case when $p = q$, Corollary 2 was proven by Hegarty [21] (without a lower bound on n).

1.2. The number of solution-free sets. Write $f(n, \mathcal{L})$ for the number of \mathcal{L} -free subsets of $[n]$. In the case when \mathcal{L} is $x + y = z$, define $f(n) := f(n, \mathcal{L})$.

By considering all possible subsets of $[n]$ consisting of odd numbers, one observes that there are at least $2^{n/2}$ sum-free subsets of $[n]$. Cameron and Erdős [11] conjectured that in fact $f(n) = \Theta(2^{n/2})$. This conjecture was proven independently by Green [17] and Sapozhenko [30]. In fact, they showed that there are constants C_1 and C_2 such that $f(n) = (C_i + o(1))2^{n/2}$ for all $n \equiv i \pmod{2}$.

Results from [23, 32] imply that there are between $2^{(1.16+o(1))\sqrt{n}}$ and $2^{(6.45+o(1))\sqrt{n}}$ Sidon sets in $[n]$. There are also several results concerning the number of so-called (k, ℓ) -sum-free subsets of $[n]$ (see, e.g., [8, 10, 33]).

More generally, given a linear equation \mathcal{L} , there are at least $2^{\mu_{\mathcal{L}}(n)}$ \mathcal{L} -free subsets of $[n]$. In light of the situation for sum-free sets one may ask whether, in general, $f(n, \mathcal{L}) = \Theta(2^{\mu_{\mathcal{L}}(n)})$. However, Cameron and Erdős [11] observed that this is false for translation-invariant \mathcal{L} . In particular, given such an \mathcal{L} -free set, any translation of it is also \mathcal{L} -free.

Green [18] though showed that given a homogeneous linear equation \mathcal{L} , $f(n, \mathcal{L}) = 2^{\mu_{\mathcal{L}}(n) + o(n)}$ (where here the $o(n)$ may depend on \mathcal{L}). Our next result implies that one can omit the term $o(n)$ in the exponent for certain types of linear equation \mathcal{L} .

Theorem 3. *Fix $p, q \in \mathbb{N}$ where (i) $q \geq 2$ and $p > q(3q - 2)/(2q - 2)$ or (ii) $q = 1$ and $p \geq 3$. Let \mathcal{L} denote the equation $px + qy = z$. Then*

$$f(n, \mathcal{L}) = \Theta(2^{\mu_{\mathcal{L}}(n)}).$$

1.3. The number of maximal solution-free sets. Given a linear equation \mathcal{L} , we say that $S \subseteq [n]$ is a *maximal \mathcal{L} -free subset* of $[n]$ if it is \mathcal{L} -free and it is not properly contained in another \mathcal{L} -free subset of $[n]$. Write $f_{\max}(n, \mathcal{L})$ for the number of maximal \mathcal{L} -free subsets of $[n]$. In the case when \mathcal{L} is $x + y = z$, define $f_{\max}(n) := f_{\max}(n, \mathcal{L})$.

A significant proportion of the sum-free subsets of $[n]$ lie in just two maximal sum-free sets, namely the set of odd numbers in $[n]$ and the interval $[\lfloor n/2 \rfloor + 1, n]$. This led Cameron and Erdős [12] to ask whether $f_{\max}(n) = o(f(n))$ or even $f_{\max}(n) \leq f(n)/2^{\varepsilon n}$ for some constant $\varepsilon > 0$. Łuczak and Schoen [25] answered this question in the affirmative, showing that $f_{\max}(n) \leq 2^{n/2 - 2^{-28}n}$ for sufficiently large n . Later, Wolfowitz [35] proved that $f_{\max}(n) \leq 2^{3n/8 + o(n)}$. Very recently, Balogh, Liu, Sharifzadeh and Treglown [2, 3] proved the following: For each $1 \leq i \leq 4$, there is a constant C_i such that, given any $n \equiv i \pmod{4}$, $f_{\max}(n) = (C_i + o(1))2^{n/4}$.

Except for sum-free sets, the problem of determining the number of maximal solution-free subsets of $[n]$ remains wide open. In this paper we give a number of bounds on $f_{\max}(n, \mathcal{L})$ for homogeneous linear equations \mathcal{L} in three variables. The next result gives a general upper bound for such \mathcal{L} . Given a three-variable linear equation \mathcal{L} , an \mathcal{L} -triple is a multiset $\{x, y, z\}$ which forms a solution to \mathcal{L} . Let $\mu_{\mathcal{L}}^*(n)$ denote the number of elements $x \in [n]$ that do not lie in *any* \mathcal{L} -triple in $[n]$.

Theorem 4. *Let \mathcal{L} be a fixed homogenous three-variable linear equation. Then*

$$f_{\max}(n, \mathcal{L}) \leq 3^{(\mu_{\mathcal{L}}(n) - \mu_{\mathcal{L}}^*(n))/3 + o(n)}.$$

Theorem 4 together with the aforementioned result of Green shows that $f_{\max}(n, \mathcal{L})$ is significantly smaller than $f(n, \mathcal{L})$ for all homogeneous three-variable linear equations \mathcal{L} that are not translation-invariant. So in this sense it can be viewed as a generalisation of the result of Łuczak and Schoen. The proof of Theorem 4 is a simple application of container and removal lemmas of Green [18]. The same idea was used to prove results in [5, 2, 3]. Although at first sight the bound in Theorem 4 may seem crude, perhaps surprisingly there are equations \mathcal{L} where the value of $f_{\max}(n, \mathcal{L})$ is close to this bound (see Proposition 21 in Section 5).

On the other hand, the following result shows that there are linear equations where the bound in Theorem 4 is far from tight.

Theorem 5. *Let \mathcal{L} denote the equation $px + qy = z$ where $p \geq q \geq 2$ are integers so that $p \leq q^2 - q$ and $\gcd(p, q) = q$. Then*

$$f_{\max}(n, \mathcal{L}) \leq 2^{(\mu_{\mathcal{L}}(n) - \mu_{\mathcal{L}}^*(n))/2 + o(n)}.$$

In the case when \mathcal{L} is the equation $2x + 2y = z$ we provide a matching lower bound. Again though, we suspect there are equations \mathcal{L} where the bound in Theorem 5 is far from tight. The proof of Theorem 5 applies Theorem 1 as well as the container and removal lemmas of Green [18].

We also provide another upper bound on $f_{\max}(n, \mathcal{L})$ for a more general class of linear equations.

Theorem 6. *Let \mathcal{L} denote the equation $px + qy = z$ where $p \geq q$, $p \geq 2$ and $p, q \in \mathbb{N}$. Then*

$$f_{\max}(n, \mathcal{L}) \leq 2^{\mu_{\mathcal{L}}(\lfloor \frac{n-p}{q} \rfloor) + o(n)}.$$

Further, if $q \geq 2$ and $p > q(3q - 2)/(2q - 2)$ or $q = 1$ and $p \geq 3$ then

$$f_{\max}(n, \mathcal{L}) = O(2^{\mu_{\mathcal{L}}(\lfloor \frac{n-p}{q} \rfloor)}).$$

In Section 5 we discuss in what cases a bound as in Theorem 6 is stronger than the bound in Theorem 5 (and vice versa). We also provide lower bounds on $f_{\max}(n, \mathcal{L})$ for all equations \mathcal{L} of the form $px + qy = z$ where $p, q \geq 2$ are integers; see Proposition 24.

Our results suggest that, in contrast to the case of $f(n, \mathcal{L})$, it is unlikely there is a ‘simple’ general asymptotic formula for $f_{\max}(n, \mathcal{L})$ for all homogeneous linear equations \mathcal{L} . It would be extremely interesting to make further progress on this problem.

The paper is organised as follows. In the next section we collect together a number of useful tools. In Section 3 we prove Theorem 1. Theorem 3 is proven in Section 4. We prove our results on the number of maximal \mathcal{L} -free sets in Section 5.

2. CONTAINERS AND INDEPENDENT SETS IN GRAPHS

2.1. Container and removal lemmas. Recently the method of *containers* has proven powerful in tackling a range of problems in combinatorics and other areas, in particular due to the work of Balogh, Morris and Samotij [4] and Saxton and Thomason [31]. Roughly speaking this method states that for certain (hyper)graphs G , the independent sets of G lie only in a small number of subsets of $V(G)$ called *containers*, where each container is an ‘almost independent set’.

Recall that, given a three-variable linear equation \mathcal{L} , an \mathcal{L} -triple is a multiset $\{x, y, z\}$ which forms a solution to \mathcal{L} . Let H denote the hypergraph with vertex set $[n]$ and edges corresponding to \mathcal{L} -triples. Then an independent set in H is precisely an \mathcal{L} -free set.

The following container lemma is a special case of a result of Green (Proposition 9.1 of [18]). Lemma 7(i)–(iii) is stated explicitly in [18]. Lemma 7(iv) follows as an immediate consequence of Lemma 7(i) and Lemma 8 below.

Lemma 7. [18] *Fix a three-variable homogeneous linear equation \mathcal{L} . There exists a family \mathcal{F} of subsets of $[n]$ with the following properties:*

- (i) *Every $F \in \mathcal{F}$ has at most $o(n^2)$ \mathcal{L} -triples.*
- (ii) *If $S \subseteq [n]$ is \mathcal{L} -free, then S is a subset of some $F \in \mathcal{F}$.*
- (iii) *$|\mathcal{F}| = 2^{o(n)}$.*
- (iv) *Every $F \in \mathcal{F}$ has size at most $\mu_{\mathcal{L}}(n) + o(n)$.*

Throughout the paper we refer to the elements of \mathcal{F} as *containers*. Notice that Lemma 7(iv) gives a bound on the size of the containers in terms of $\mu_{\mathcal{L}}(n)$ even though, in general, the precise value of $\mu_{\mathcal{L}}(n)$ is not known.

The following removal lemma is a special case of a result of Green (Theorem 1.5 in [18]). This result was also generalised to systems of linear equations by Král’, Serra and Vena (Theorem 2 in [24]).

Lemma 8. [18] *Fix a three-variable homogeneous linear equation \mathcal{L} . Suppose that $A \subseteq [n]$ is a set containing $o(n^2)$ \mathcal{L} -triples. Then there exist B and C such that $A = B \cup C$ where B is \mathcal{L} -free and $|C| = o(n)$.*

We will also apply the following bound on the number of \mathcal{L} -free sets.

Theorem 9. [18] *Fix a homogeneous linear equation \mathcal{L} . Then $f(n, \mathcal{L}) = 2^{\mu_{\mathcal{L}}(n) + o(n)}$.*

We will use the above results to deduce upper bounds on the number of maximal \mathcal{L} -free sets (Theorems 4, 5 and 6).

2.2. Independent sets in graphs. Let G be a graph and consider any subset $X \subseteq V(G)$. Let $\text{IS}(G)$ denote the number of independent sets in G . Let $G[X]$ denote the induced subgraph of G on the vertex set X and $G \setminus X$ denote the induced subgraph of G on the vertex set $V(G) \setminus X$.

Fact 10. *Let G be a graph and let A_1, \dots, A_r be a partition of $V(G)$. Then $\text{IS}(G) \leq \prod_{i=1}^r \text{IS}(G[A_i])$.*

The following simple lemma will be used in the proof of Theorem 3.

Lemma 11. *Let G be a graph on n vertices and M be a matching in G which consists of e edges. Suppose that $v \in V(G)$ lies in M . Then the number of independent sets in G which contain v is at most $3^{e-1} \cdot 2^{n-2e}$.*

Proof. First note that the number of independent sets in G which contain v is at most $\text{IS}(G \setminus X)$ where X consists of v and its neighbour in M . Let A_1, \dots, A_e be a partition of the vertex set $V(G \setminus X)$, where if $1 \leq i \leq e-1$ then A_i contains precisely the two vertices from some edge in M . So $|A_e| = n - 2e$. Clearly $\text{IS}(G[A_i]) = 3$ for $1 \leq i \leq e-1$ and $\text{IS}(G[A_e]) \leq 2^{n-2e}$. The result then follows by Fact 10. \square

2.3. Link graphs and maximal independent sets. We obtain many of our results by counting the number of maximal independent sets in various auxiliary graphs. Similar techniques were used in [35, 2, 3], and in the graph setting in [5, 1]. To be more precise, let B and S be disjoint subsets of $[n]$ and fix a three-variable linear equation \mathcal{L} . The *link graph* $L_S[B]$ of S on B has vertex set B , and an edge set consisting of the following two types of edges:

- (i) Two vertices x and y are adjacent if there exists an element $z \in S$ such that $\{x, y, z\}$ is an \mathcal{L} -triple;
- (ii) There is a loop at a vertex x if there exists an element $z \in S$ or elements $z, z' \in S$ such that $\{x, x, z\}$ or $\{x, z, z'\}$ is an \mathcal{L} -triple.

Notice that since the only possible trivial solutions to a three-variable linear equation \mathcal{L} are of the form $\{x, x, x\}$, all the edges in $L_S[B]$ correspond to non-trivial \mathcal{L} -triples.

The following simple lemma was stated in [2, 3] for sum-free sets, but extends to three-variable linear equations.

Lemma 12. *Fix a three-variable linear equation \mathcal{L} . Suppose that B, S are disjoint \mathcal{L} -free subsets of $[n]$. If $I \subseteq B$ is such that $S \cup I$ is a maximal \mathcal{L} -free subset of $[n]$, then I is a maximal independent set in $G := L_S[B]$.*

Let $\text{MIS}(G)$ denote the number of maximal independent sets in G . Suppose we have a container $F \in \mathcal{F}$ as in Lemma 7 and suppose $F = A \cup B$ where B is \mathcal{L} -free. Observe that any maximal \mathcal{L} -free subset of $[n]$ in F can be found by first choosing an \mathcal{L} -free set $S \subseteq A$, and then extending S in B . Note that by Lemma 12, the number of possible extensions of S in B (which we shall refer to as $N(S, B)$) is bounded from above by the number of maximal independent sets in the link graph $L_S[B]$ (i.e. we have $N(S, B) \leq \text{MIS}(L_S[B])$). Hence Lemma 12 is a useful tool for bounding the number of maximal \mathcal{L} -free subsets of $[n]$.

In particular, we will apply the following result in combination with Lemma 12. The first part was proven by Moon and Moser [26] and the second part by Hujter and Tuza [22]. We use the first condition in the proof of Theorems 4 and 5.

Theorem 13. *Suppose that G is a graph on n vertices possibly with loops. Then the following bounds hold.*

- (i) $\text{MIS}(G) \leq 3^{n/3}$;
- (ii) $\text{MIS}(G) \leq 2^{n/2}$ if G is additionally triangle-free.

To prove Theorem 5 we will combine Theorem 13(ii) and the following result.

Lemma 14. Let \mathcal{L} denote the equation $px + qy = z$ where $p \geq q \geq 2$ and $p, q \in \mathbb{N}$. Let $A \subseteq [1, u]$ and let $B \subseteq [u+1, n]$ for some $u \in [n]$. Consider the link graph $G := L_A[B]$ of A on B . If $q^2 \geq p+q$ then G is triangle-free.

Proof. Suppose that $q^2 \geq p+q$ and suppose for a contradiction there is a triangle in G with vertices $b_1 < b_2 < b_3$. By definition of the link graph, there exist $s_1, s_2, s_3 \in A$ such that $\{b_1, b_2, s_1\}, \{b_2, b_3, s_2\}, \{b_1, b_3, s_3\}$ are \mathcal{L} -triples.

Since all numbers in A are smaller than all numbers in B we have $1 \leq s_1, s_2, s_3 < b_1 < b_2 < b_3$. Also, since $p \geq q \geq 2$, for each of our \mathcal{L} -triples $\{b_i, b_j, s_k\}$ (where $b_i < b_j$) it follows that b_j must play the role of z in \mathcal{L} .

Define a multiset $\{r_i \in \{p, q\} : 1 \leq i \leq 6, r_1 \neq r_2, r_3 \neq r_4, r_5 \neq r_6\}$. Consider the three equations $r_1 b_1 + r_2 s_1 = b_2, r_3 b_2 + r_4 s_2 = b_3$ and $r_5 b_1 + r_6 s_3 = b_3$. Combining the second and third gives $b_2 = (r_5 b_1 + r_6 s_3 - r_4 s_2)/r_3$. Then combining this with the first equation gives $(r_1 r_3 - r_5) b_1 + r_2 r_3 s_1 + r_4 s_2 = r_6 s_3$. Now since $s_3 < b_1$ and all terms are at least 1, for such an inequality to hold we must have $r_1 r_3 - r_5 < r_6$. Since $r_5 \neq r_6$ this means we have $r_1 r_3 < p+q$. Hence as $r_1, r_3 \in \{p, q\}$, in order for G to have a triangle at least one of $p^2 < p+q, q^2 < p+q$ and $pq < p+q$ must be satisfied. Since $p \geq q \geq 2$, the first and third are not true and so we must have $q^2 < p+q$, a contradiction. \square

We also use link graphs as a means to obtain lower bounds on the number of maximal \mathcal{L} -free sets. We apply the following result in Propositions 21 and 24.

Lemma 15. Fix a three-variable linear equation \mathcal{L} . Suppose that B, S are disjoint \mathcal{L} -free subsets of $[n]$. Let H be an induced subgraph of the link graph $L_S[B]$. Then $f_{\max}(n, \mathcal{L}) \geq \text{MIS}(H)$.

Proof. Suppose I and J are different maximal independent sets in H . First note that $S \cup I$ and $S \cup J$ are \mathcal{L} -free by definition of the link graph. Both cannot lie in the same maximal \mathcal{L} -free subset of $[n]$. To see this, observe by definition of I and J , there exists $i \in I \setminus J$. There must exist $s \in S, j \in J$ such that $\{i, j, s\}$ forms an \mathcal{L} -triple, else $J \cup \{i\}$ would be an independent set in H , which contradicts the maximality of J . Hence any maximal \mathcal{L} -free subset of $[n]$ containing $S \cup J$ does not contain i . Similarly there exists $j \in J \setminus I$ such that any maximal \mathcal{L} -free subset of $[n]$ containing $S \cup I$ does not contain j . The result immediately follows. \square

3. THE SIZE OF THE LARGEST SOLUTION-FREE SET

Throughout this section, \mathcal{L} will denote the equation $px + qy = z$ where $p \geq q$ and $p \geq 2, p, q \in \mathbb{N}$. The aim of this section is to determine the size of the largest \mathcal{L} -free subset of $[n]$. In fact, we will prove a richer structural result on \mathcal{L} -free sets (Theorem 1). For this, we will introduce the following auxiliary graph G_m : Let $m \in [n]$ be fixed. We define the graph G_m to have vertex set $[m, n]$ and edges between c and $pm + qc$ for all $c \in [m, n]$ such that $pm + qc \leq n$. We will also make use of these auxiliary graphs in Section 4.

Fact 16.

- (i) The size of the largest \mathcal{L} -free subset S of $[n]$ with $\min(S) = m$ is at most the size of the largest independent set in G_m which contains m .
- (ii) The number of \mathcal{L} -free subsets S of $[n]$ with $\min(S) = m$ is at most the number of independent sets in G_m which contain m .

Proof. Let S be an \mathcal{L} -free subset of $[n]$ with $\min(S) = m$. Since $\{m, c, pm + qc\}$ is an \mathcal{L} -triple contained in $[n]$ for all $c \in [m, n]$ such that $pm + qc \leq n$, S cannot contain both c and $pm + qc$. Hence any \mathcal{L} -free subset of $[n]$ with minimum element m is also an independent set in G_m which

contains m (although the converse does not necessarily hold). This immediately implies (i) and (ii). \square

Note that G_m is a union of disjoint paths (possibly isolated vertices). We refer to the connected components of G_m as the *path components*. Given G_m , we define $y_0 := n$, and for $i \geq 1$ define $y_i := \max\{v \in V(G_m) \mid pm + qv \leq y_{i-1}\}$. Thus we have $y_i = \lfloor \frac{y_{i-1} - pm}{q} \rfloor$. For G_m we also define k to be the largest i such that $y_i \in [m, n]$, and refer to k as the *path parameter* of G_m . We define the *size* of a path component to be the number of *vertices* in it, and we define $N(G_m, i)$ to be the number of path components of size i in G_m .

Fact 17. *The graph G_m consists entirely of disjoint path components, where for each $1 \leq i \leq k-1$ there are $y_{i-1} + y_{i+1} - 2y_i$ path components of size i , there are $y_{k-1} - 2y_k + m - 1$ path components of size k and $y_k - m + 1$ path components of size $k+1$.*

Proof. Every vertex $c \in V(G_m)$ satisfying $y_{j+1} < c \leq y_j$ for some $0 \leq j \leq k-1$ is in a path in G_m which contains precisely j vertices which are larger than it, whereas every vertex $c > y_j$ is not in such a path. All the vertices in $[m, y_k]$ are in paths which contain precisely k vertices which are larger than it, all vertices in $[y_k + 1, y_{k-1}]$ are in paths which contain precisely $k-1$ vertices which are larger than it, and so on.

Let A_i be the interval $[y_i + 1, y_{i-1}]$ for $1 \leq i \leq k$ and let A_{k+1} be the interval $[m, y_k]$. There are $|[m, y_k]| = y_k - m + 1$ path components of size $k+1$ in G_m . For $i \leq k$ all vertices in A_i are the smallest vertex in a path on i vertices, however they may not be the smallest vertex in their path component. In fact, by definition of the y_i , all paths which start in A_j for some j must include precisely one vertex from each set $A_{j-1}, A_{j-2}, \dots, A_1$. This means that for $i \leq k$, the number of path components of size i in G_m is precisely $|A_i| - |A_{i+1}|$. For $i \leq k-1$ this is $y_{i-1} + y_{i+1} - 2y_i$ and for $i = k$ this is $y_{k-1} - 2y_k + m - 1$. \square

We now use the graphs G_m and the above facts to obtain the bound for the size of the largest \mathcal{L} -free subset of $[n]$ as stated in Theorem 1.

Proof of Theorem 1. Let t be a non-negative integer. To prove (i) suppose that $t < (\frac{p+q-1}{p+q+p/q}) \lfloor \frac{n}{p+q} \rfloor$. Suppose S is an \mathcal{L} -free set contained in $[\lfloor \frac{n}{p+q} \rfloor - t, n]$ where $m := \lfloor \frac{n}{p+q} \rfloor - t \in S$. By Fact 16(i) we wish to prove that the largest independent set in G_m containing m has size at most $\lceil \frac{(p+q-1)n}{p+q} \rceil - \lfloor \frac{p}{q}t \rfloor$. Since $|V(G_m)| = \lceil \frac{(p+q-1)n}{p+q} \rceil + t + 1$ it suffices to show that any independent set I in G_m satisfies $|V(G_m) \setminus I| \geq \lfloor (p+q)t/q \rfloor + 1$.

For $0 \leq i \leq \lfloor (p+q)t/q \rfloor$, there is an edge between $m+i$ and $(p+q)m+qi$. Note that since $i \leq \lfloor (p+q)t/q \rfloor$ and $q \leq p$ we have that the largest vertex in any of these edges is indeed at most n :

$$(p+q)(\lfloor \frac{n}{p+q} \rfloor - t) + qi \leq n - (p+q)t + q\lfloor (p+q)t/q \rfloor \leq n - (p+q)t + q(p+q)t/q = n.$$

Since I can only contain one vertex from each of these edges, we have proven (i), provided that these edges are disjoint. It suffices to show that $\lfloor \frac{n}{p+q} \rfloor + \lfloor pt/q \rfloor < (p+q)m = (p+q)(\lfloor \frac{n}{p+q} \rfloor - t)$ since the left hand side is the largest element of the set $\{m+i : 0 \leq i \leq \lfloor (p+q)t/q \rfloor\}$. But this immediately follows since $t < (\frac{p+q-1}{p+q+p/q}) \lfloor \frac{n}{p+q} \rfloor$.

To prove (ii) let $t \geq (\frac{p+q-1}{p+q+p/q}) \lfloor \frac{n}{p+q} \rfloor$ and suppose S is an \mathcal{L} -free subset of $[n]$ with $m := \min(S) = \lfloor \frac{n}{p+q} \rfloor - t$. By Fact 16(i) $|S|$ is at most the size of the largest independent set in G_m which contains m . We will first show that G_m has path parameter $k \geq 2$, and then the case $q = 1$ follows easily.

Define $\ell := \lfloor k/2 \rfloor$ and

$$C_k := \left(\frac{\sum_{i=0}^{2\ell+1} (-1)(-q)^i + p \sum_{i=0}^{\ell} q^{2i}}{q^{2\ell+1} + p \sum_{i=0}^{2\ell} q^i} \right).$$

We will show that if $q \geq 2$ then the largest independent set in G_m has size at most $C_k n + k$. We then further bound this from above by $(q^2 + 1)n/(q^2 + q + 1)$ for n sufficiently large.

Note that by Fact 17, to prove that $k \geq 2$ for G_m it suffices to show that there is a path on 3 vertices in G_m . By definition of k , m lies on a path P on $k + 1$ vertices. Write $P = v_0 v_1 \cdots v_k$ where $m = v_0$ and observe that $v_j = (q^j + p \sum_{i=0}^{j-1} q^i)m$ for $0 \leq j \leq k$. To prove $k \geq 2$ it suffices to show that there is indeed a vertex $(q^2 + pq + p)m$ in $V(G_m)$, i.e. $(q^2 + pq + p)m \leq n$. Note that since $t \geq (\frac{p+q-1}{p+q+p/q}) \lfloor \frac{n}{p+q} \rfloor$, we have $m = \lfloor \frac{n}{p+q} \rfloor - t \leq (\frac{p+q+p/q-p-q+1}{p+q+p/q}) \lfloor \frac{n}{p+q} \rfloor = (\frac{p+q}{q^2+pq+p}) \lfloor \frac{n}{p+q} \rfloor$. Hence $(q^2 + pq + p)m \leq n$ as desired.

When $q = 1$ observe that $y_i = y_{i-1} - pm$, so for $i \leq k - 1$ by Fact 17 we have $N(G_m, i) = y_{i-1} + y_{i+1} - 2y_i = y_i + pm + y_i - pm - 2y_i = 0$. Hence G_m consists entirely of a union of path components of size either k or $k + 1$. Since at most $\lceil i/2 \rceil$ vertices of a path on i vertices can be in an independent set and $k \geq 2$, the largest independent set in G_m has size at most $2n/3 = (q^2 + 1)n/(q^2 + q + 1)$ in this case, as desired. So now consider the case when $q \geq 2$. We calculate the maximum size of an independent set in G_m :

$$\begin{aligned} & \sum_{i=1}^{k+1} \lceil i/2 \rceil \cdot N(G_m, i) \\ &= \left(\sum_{i=1}^{k-1} \lceil i/2 \rceil \cdot (y_{i-1} + y_{i+1} - 2y_i) \right) + \lceil k/2 \rceil (y_{k-1} + m - 1 - 2y_k) + \lceil (k+1)/2 \rceil (y_k - m + 1) \\ (2) \quad &= y_0 + \left(\sum_{i=1}^k (\lceil (i-1)/2 \rceil - 2\lceil i/2 \rceil + \lceil (i+1)/2 \rceil) y_i \right) + (m-1)(\lceil k/2 \rceil - \lceil (k+1)/2 \rceil). \end{aligned}$$

Here we used Fact 17 in the first equality. For i odd, the coefficient of y_i in (2) is $(i-1)/2 - 2(i+1)/2 + (i+1)/2 = -1$. For i even, the coefficient of y_i in (2) is $i/2 - 2i/2 + (i+2)/2 = 1$.

The following bounds are obtained from the definition of y_i and k :

$$\begin{aligned} (a) \quad & \left(n - q^j + 1 - pm \sum_{i=0}^{j-1} q^i \right) / q^j \leq y_j \leq \left(n - pm \sum_{i=0}^{j-1} q^i \right) / q^j; \\ (b) \quad & n / \left(q^{k+1} + p \sum_{i=0}^k q^i \right) < m \leq n / \left(q^k + p \sum_{i=0}^{k-1} q^i \right). \end{aligned}$$

Let $\ell := \lfloor k/2 \rfloor$ (note $k \geq 2$ so $\ell \geq 1$). First suppose k is odd, i.e. $k = 2\ell + 1$. Using (2), the size of the largest independent set in G_m is bounded above by

$$\begin{aligned}
& y_0 + \left(\sum_{i=1}^k (\lceil (i-1)/2 \rceil - 2\lceil i/2 \rceil + \lceil (i+1)/2 \rceil) y_i \right) + (m-1)(\lceil k/2 \rceil - \lceil (k+1)/2 \rceil) \\
&= y_0 - y_1 + y_2 - y_3 + \cdots + y_{2\ell} - y_{2\ell+1} \\
&\stackrel{(a)}{\leq} n - \left(\frac{n - pm - q + 1}{q} \right) + \left(\frac{n - pm(1+q)}{q^2} \right) - \left(\frac{n - pm(1+q+q^2) - q^3 + 1}{q^3} \right) \\
&\quad + \cdots - \left(\frac{n - \left(pm \sum_{i=0}^{2\ell} q^i \right) - q^{2\ell+1} + 1}{q^{2\ell+1}} \right) \\
&= n \left(1 - \frac{1}{q} + \frac{1}{q^2} - \cdots - \frac{1}{q^{2\ell+1}} \right) + m \left(\frac{p}{q} + \frac{p}{q^3} + \cdots + \frac{p}{q^{2\ell+1}} \right) + \frac{q-1}{q} + \frac{q^3-1}{q^3} + \cdots + \frac{q^{2\ell+1}-1}{q^{2\ell+1}} \\
&\stackrel{(b)}{\leq} \frac{n}{q^{2\ell+1}} \left(\sum_{i=0}^{2\ell+1} (-1)(-q)^i \right) + \left(\frac{n}{q^{2\ell+1} + p \sum_{i=0}^{2\ell} q^i} \right) \left(\frac{p \sum_{i=0}^{\ell} q^{2i}}{q^{2\ell+1}} \right) + \frac{k+1}{2} \\
&= \left(\frac{\left[\sum_{i=0}^{2\ell+1} (-1)(-q)^i \right] (q^{2\ell+1} + p \sum_{i=0}^{2\ell} q^i) + p \sum_{i=0}^{\ell} q^{2i}}{q^{2\ell+1} (q^{2\ell+1} + p \sum_{i=0}^{2\ell} q^i)} \right) n + \frac{k+1}{2} \\
&= \left(\frac{\sum_{i=0}^{2\ell+1} (-q)^{i+2\ell+1} + p \sum_{i=0}^{\ell} q^{2i+2\ell+1}}{q^{2\ell+1} (q^{2\ell+1} + p \sum_{i=0}^{2\ell} q^i)} \right) n + \frac{k+1}{2} = \left(\frac{\sum_{i=0}^{2\ell+1} (-1)(-q)^i + p \sum_{i=0}^{\ell} q^{2i}}{q^{2\ell+1} + p \sum_{i=0}^{2\ell} q^i} \right) n + \frac{k+1}{2} \\
&= C_k n + \frac{k+1}{2} \leq C_k n + k.
\end{aligned}$$

(Note that some of our calculations above did indeed require $q \geq 2$.) By definition, $m \geq y_{k+1} + 1$ and for k even, we have $C_k = C_{k+1}$. So if k is even ($k = 2\ell$) then we have

$$\begin{aligned}
& y_0 + \left(\sum_{i=1}^k (\lceil (i-1)/2 \rceil - 2\lceil i/2 \rceil + \lceil (i+1)/2 \rceil) y_i \right) + (m-1)(\lceil k/2 \rceil - \lceil (k+1)/2 \rceil) \\
&= y_0 - y_1 + y_2 - y_3 + \cdots + y_{2\ell} - m + 1 \leq y_0 - y_1 + y_2 - y_3 + \cdots + y_{2\ell} - y_{2\ell+1} \\
&\leq C_{k+1} n + \frac{k+2}{2} \leq C_k n + k.
\end{aligned}$$

The penultimate inequality follows by using calculations from the odd case. The last inequality follows since $k \geq 2$ and $C_k = C_{k+1}$. Thus we have shown that $|S| \leq C_k n + k$ and we know that $k \geq 2$. It remains to show that

$$(3) \quad C_k n + k \leq \frac{(q^2 + 1)n}{q^2 + q + 1}$$

for $k \geq 2$ and n sufficiently large.

We know that $m \leq n/(q^k + p \sum_{i=0}^{k-1} q^i)$ and so $n \geq q^k + p \sum_{i=0}^{k-1} q^i$, therefore condition (3) is met if

$$(4) \quad \left(\frac{q^2 + 1}{q^2 + q + 1} - C_k \right) \left(q^k + p \sum_{i=0}^{k-1} q^i \right) \geq k.$$

Claim 18. For $k \geq 6$, (4) holds.

Since the proof of Claim 18 is just a technical calculation, we defer it to the appendix.

The claim is not a result which generally holds for $2 \leq k \leq 5$ so instead we directly calculate how large n should be to satisfy (3) in these cases. For $k = 3$ and $k = 5$ we obtain $n \geq \frac{3(q^3 + p(q^2 + q + 1))(q^2 + q + 1)}{q^2 + 1}$ and $n \geq \frac{5(q^5 + p(q^4 + q^3 + q^2 + q + 1))(q^2 + q + 1)}{q^4 + (p-1)q^3 + q^2 + 1}$ respectively. For $k = 2$ and $k = 4$ we obtain weaker bounds. Hence taking n to be sufficiently large (larger than these two bounds), we have $C_k n + k \leq \frac{(q^2 + 1)n}{q^2 + q + 1}$ for all $k \geq 2$. □

4. THE NUMBER OF SOLUTION-FREE SETS

Recall a theorem of Green [18] states that $f(n, \mathcal{L}) = 2^{\mu_{\mathcal{L}}(n) + o(n)}$ for any fixed homogeneous linear equation \mathcal{L} . The aim of this section is to replace the term $o(n)$ here with a constant for many equations \mathcal{L} . This will be achieved in Theorem 20, which immediately implies Theorem 3. Denote by $f(n, \mathcal{L}, m)$ the number of \mathcal{L} -free subsets of $[n]$ with minimum element m . We first give bounds on $f(n, \mathcal{L}, m)$ for linear equations \mathcal{L} of the form $px + qy = z$.

Lemma 19. Let \mathcal{L} denote the equation $px + qy = z$ where $p \geq q$ and $p \geq 2, p, q \in \mathbb{N}$.

- (i) If $m \geq \lfloor \frac{n}{p+q} \rfloor + 1$ then $f(n, \mathcal{L}, m) = 2^{n-m}$.
- (ii) If $m = \lfloor \frac{n}{p+q} \rfloor$ then $f(n, \mathcal{L}, m) \leq 2^{\mu_{\mathcal{L}}(n)-1}$.
- (iii) If $q \geq 2, m = \lfloor \frac{n}{p+q} \rfloor - t$ for some positive integer t and G_m has path parameter 1, then $f(n, \mathcal{L}, m) \leq 2^{\mu_{\mathcal{L}}(n) - 3/5 + t(3q-2p)/(5q)}$.
- (iv) If $q \geq 2, m = \lfloor \frac{n}{p+q} \rfloor - t$ for some positive integer t and G_m has path parameter $k \geq 2$, then $f(n, \mathcal{L}, m) \leq (4/3) \cdot 2^{(5q^2 - 2q + 2)n/(5q^2)}$.
- (v) If $q = 1, G_m$ has path parameter ℓ , and $m = \lfloor \frac{n}{\ell p + 1} \rfloor - t$ for some integer t , then $f(n, \mathcal{L}, m) \leq 2^{(7\ell p + 3p)n/(10\ell p + 10) + t(7-3p)/10}$.

Proof. First note that (i) is trivial since all subsets $S \subseteq [n]$ with $\min(S) \geq \lfloor \frac{n}{p+q} \rfloor + 1$ are \mathcal{L} -free. By Fact 16(ii) we know that $f(n, \mathcal{L}, m)$ is at most the number of independent sets in G_m which contain m . For (ii), there is one edge between $m = \lfloor \frac{n}{p+q} \rfloor$ and $(p+q)m \leq n$ in G_m , hence there are at most $2^{n - \lfloor \frac{n}{p+q} \rfloor - 1} = 2^{\mu_{\mathcal{L}}(n) - 1}$ independent sets in G_m containing m .

For (iii) suppose $q \geq 2$ and $m = \lfloor \frac{n}{p+q} \rfloor - t$ for some $t \in \mathbb{N}$. Notice that G_m contains a matching on $y_1 - m + 1$ edges, namely there is an edge between c and $pm + qc$ for $c \in [m, y_1]$. Observe that $3/4 \leq 2^{-2/5}$ and also

$$y_1 - m = \left\lfloor \frac{n - pm}{q} \right\rfloor - m \geq \frac{n - (p+q)m - q}{q} \geq \frac{t(p+q)}{q} - 1.$$

Hence by Lemma 11 the total number of independent sets in G_m which contain m is at most

$$\begin{aligned} 2^{n-m-2(y_1-m)-1} 3^{y_1-m} &\leq 2^{\mu_{\mathcal{L}}(n)-1+t} (3/4)^{y_1-m} \\ &\leq 2^{\mu_{\mathcal{L}}(n)-1+t} (3/4)^{t(p+q)/q-1} \leq 2^{\mu_{\mathcal{L}}(n)-3/5+t(3q-2p)/(5q)}, \end{aligned}$$

as desired.

For (iv) suppose $q \geq 2$, $m = \lfloor \frac{n}{p+q} \rfloor - t$ for some positive integer t and G_m has path parameter $k \geq 2$. First note that

$$\begin{aligned} y_1 - y_2 &= \left\lfloor \frac{n - pm}{q} \right\rfloor - \left\lfloor \frac{\lfloor \frac{n - pm}{q} \rfloor - pm}{q} \right\rfloor \geq \frac{n - pm - q}{q} - \frac{n - pm - qpm}{q^2} \\ &= \frac{(q - 1)n + pm - q^2}{q^2} \geq \frac{(q - 1)n}{q^2} - 1. \end{aligned}$$

Define $F(i)$ to be the i th Fibonacci number where $F(1) = F(2) = 1$. There are $F(i + 2)$ independent sets (including the empty set) in a path of length i . Recall the following Fibonacci identity: $F(i + 2)F(i) - F(i + 1)^2 = (-1)^{i+1}$. If i is even and $a > b$ then

$$\left(\frac{F(i)F(i + 2)}{F(i + 1)^2} \right)^a \left(\frac{F(i + 1)F(i + 3)}{F(i + 2)^2} \right)^b = \left(\frac{F(i + 1)^2 - 1}{F(i + 1)^2} \right)^a \left(\frac{F(i + 2)^2 + 1}{F(i + 2)^2} \right)^b \leq 1.$$

Also observe that by omitting $(F(i + 1)F(i + 3)/F(i + 2)^2)^b$ the inequality still holds. By use of Fact 17 and applying the above bounds, we can bound from above the number of independent sets in G_m as required:

$$\begin{aligned} &2^{y_0 + y_2 - 2y_1} 3^{y_1 + y_3 - 2y_2} 5^{y_2 + y_4 - 2y_3} \dots F(k + 1)^{y_{k-2} + y_k - 2y_{k-1}} F(k + 2)^{y_{k-1} + m - 2y_k - 1} F(k + 3)^{y_k - m + 1} \\ &= 2^{y_0 + y_2 - 2y_1} 3^{y_1 - 2y_2} 5^{y_2} \left(\frac{3 \cdot 8}{5^2} \right)^{y_3} \left(\frac{5 \cdot 13}{8^2} \right)^{y_4} \dots \left(\frac{F(k + 1) \cdot F(k + 3)}{F(k + 2)^2} \right)^{y_k} \left(\frac{F(k + 2)}{F(k + 3)} \right)^{m-1} \\ &\leq 2^{y_0 + y_2 - 2y_1} 3^{y_1 - 2y_2} 5^{y_2} \leq 2^{y_0 + y_2 - 2y_1 + y_2} 3^{y_1 - y_2} = 2^{y_0} (3/4)^{y_1 - y_2} \leq 2^n (3/4)^{(q-1)n/q^2 - 1} \\ &\leq (4/3) \cdot 2^{n - 2(q-1)n/(5q^2)} = (4/3) \cdot 2^{(5q^2 - 2q + 2)n/(5q^2)}. \end{aligned}$$

For (v), since $y_i = n - ipm$ Fact 17 implies that if G_m has path parameter ℓ , then G_m is a union of paths of length ℓ and $\ell + 1$. We use the bound $F(i) \leq 2^{(7i-11)/10}$ (a simple proof by induction which holds for $i \geq 2$). Since $m < y_\ell = n - \ell pm$ we can write $m = \lfloor \frac{n}{\ell p + 1} \rfloor - t$ for some integer $t \geq 0$. Now using these bounds, we have

$$\begin{aligned} &F(\ell + 2)^{y_{\ell-1} - 2y_\ell + m} F(\ell + 3)^{y_\ell - m} = F(\ell + 2)^{(\ell p + p + 1)m - n} F(\ell + 3)^{n - (\ell p + 1)m} \\ &\leq 2^{(3+7\ell)((\ell p + p + 1)m - n)/10 + (10+7\ell)(n - (\ell p + 1)m)/10} = 2^{(7n + (3p-7)m)/10} \\ &\leq 2^{(7n + (3p-7)(n/(\ell p + 1) - t))/10} = 2^{(7\ell p + 3p)n/(10\ell p + 10) + t(7-3p)/10}. \end{aligned}$$

□

Theorem 20. Let \mathcal{L} denote the equation $px + qy = z$ where $p, q \in \mathbb{N}$ and

- (i) $q \geq 2$ and $p > q(3q - 2)/(2q - 2)$ or;
- (ii) $q = 1$ and $p \geq 3$.

Then $f(n, \mathcal{L}) \leq (3/2 + o(1) + C)2^{\mu_{\mathcal{L}}(n)}$ where for (i) $C := \frac{2^{-2p/(5q)}}{1 - 2^{(3q-2p)/(5q)}}$ and for (ii) $C := \frac{2^{(7-3p)/10}}{1 - 2^{(7-3p)/10}}$.

Proof. For both cases by Lemma 19(i)–(ii) there are at most $3 \cdot 2^{\mu_{\mathcal{L}}(n)-1}$ \mathcal{L} -free subsets S of $[n]$ where $\min(S) \geq \lfloor \frac{n}{p+q} \rfloor$. For (i), first consider \mathcal{L} -free subsets arising from Lemma 19(iv). Since

$k \geq 2$,

$$m < y_2 = \left\lfloor \frac{\lfloor \frac{n-pm}{q} \rfloor - pm}{q} \right\rfloor \leq \frac{n - pm - qpm}{q^2}$$

and so $m \leq n/(q^2 + pq + p)$. Now as $n \rightarrow \infty$,

$$\frac{n/(q^2 + pq + p) \cdot (4/3) \cdot 2^{(5q^2 - 2q + 2)n/(5q^2)}}{2^{\mu_{\mathcal{L}}(n)}} = \frac{2^{\log_2(4n/(3(q^2 + pq + p))) + (5q^2 - 2q + 2)n/(5q^2)}}{2^{\mu_{\mathcal{L}}(n)}} \rightarrow 0,$$

as long as we have $2^{(5q^2 - 2q + 2)n/(5q^2)} \ll 2^{\mu_{\mathcal{L}}(n)}$. This is satisfied if $(5q^2 - 2q + 2)/(5q^2) < (p + q - 1)/(p + q)$ which when rearranged, gives $p > q(3q - 2)/(2q - 2)$.

For \mathcal{L} -free subsets arising from Lemma 19(iii), set $a := 2^{\mu_{\mathcal{L}}(n) - 3/5}$, $r := 2^{(3q - 2p)/(5q)}$ and let u be the largest t such that G_m with $m = \lfloor \frac{n}{p+q} \rfloor - t$ has path parameter 1. Then since $p > q(3q - 2)/(2q - 2) > 3q/2$ we have $|r| < 1$ and so

$$\sum_{t=1}^u 2^{\mu_{\mathcal{L}}(n) - 3/5 + t(3q - 2p)/(5q)} \leq \sum_{t=1}^{\infty} ar^t = \sum_{t=0}^{\infty} (ar)r^t = \frac{ar}{1 - r} = \frac{2^{\mu_{\mathcal{L}}(n) - 2p/(5q)}}{1 - 2^{(3q - 2p)/(5q)}}.$$

Altogether this implies that $f(n, \mathcal{L}) \leq (3/2 + o(1) + C)2^{\mu_{\mathcal{L}}(n)}$ where $C := \frac{2^{-2p/(5q)}}{1 - 2^{(3q - 2p)/(5q)}}$.

For (ii), set $a := 2^{(7kp + 3p)n/(10kp + 10)}$, set $r := 2^{(7 - 3p)/10}$ and let u be the largest t such that G_m with $m := \lfloor \frac{n}{p+q} \rfloor - t$ has path parameter k for any fixed $k \in \mathbb{N}$. Since $p \geq 3$ we have $|r| < 1$ and so

$$\sum_{t=1}^u 2^{(7kp + 3p)n/(10kp + 10) + t(7 - 3p)/10} \leq \sum_{t=1}^{\infty} ar^t = \sum_{t=0}^{\infty} (ar)r^t = \frac{ar}{1 - r} = \frac{2^{(7kp + 3p)n/(10kp + 10) + (7 - 3p)/10}}{1 - 2^{(7 - 3p)/10}}.$$

For $k = 1$ the last term is at most $2^{(\mu_{\mathcal{L}}(n) + (7 - 3p)/10)/(1 - 2^{(7 - 3p)/10})}$. For $k \geq 2$ we obtain a term which is $o(2^{\mu_{\mathcal{L}}(n)})$ as n tends to infinity, since $(7kp + 3p)n/(10kp + 10) < \mu_{\mathcal{L}}(n)$ for $p \geq 3$. Therefore, Lemma 19 implies that $f(n, \mathcal{L}) \leq (3/2 + o(1) + C)2^{\mu_{\mathcal{L}}(n)}$ where $C := \frac{2^{(7 - 3p)/10}}{1 - 2^{(7 - 3p)/10}}$. \square

5. THE NUMBER OF MAXIMAL SOLUTION-FREE SETS

5.1. A general upper bound. Let \mathcal{L} be a three-variable linear equation. Let $\mathcal{M}_{\mathcal{L}}(n)$ denote the set of elements $x \in [n]$ such that $x \in [n]$ does not lie in *any* \mathcal{L} -triple in $[n]$. Define $\mu_{\mathcal{L}}^*(n) := |\mathcal{M}_{\mathcal{L}}(n)|$. For example, if \mathcal{L} is translation-invariant then $\{x, x, x\}$ is an \mathcal{L} -triple for all $x \in [n]$ so $\mathcal{M}_{\mathcal{L}}(n) = \emptyset$ and $\mu_{\mathcal{L}}^*(n) = 0$.

Let \mathcal{L} denote the equation $px + qy = z$ where $p \geq 2$, $p \geq q$ and $p, q \in \mathbb{N}$. Write $u := \gcd(p, q)$. Then notice that $\mathcal{M}_{\mathcal{L}}(n) \supseteq \{s \in [n] : s > \lfloor (n - p)/q \rfloor, u \nmid s\}$. This follows since if $s > \lfloor (n - p)/q \rfloor$ then $ps + q \geq qs + p > n$ and so s cannot play the role of x or y in an \mathcal{L} -triple in $[n]$. If $u \nmid s$ then as $u|(px + qy)$ for any $x, y \in [n]$ we have that s cannot play the role of z in an \mathcal{L} -triple in $[n]$. Actually, for large enough n we have $\mathcal{M}_{\mathcal{L}}(n) = \{s : s > \lfloor (n - p)/q \rfloor, u \nmid s\}$ for all such \mathcal{L} . We omit the proof of this here.

We now prove Theorem 4.

Theorem 4. Let \mathcal{L} be a fixed homogenous three-variable linear equation. Then

$$f_{\max}(n, \mathcal{L}) \leq 3^{(\mu_{\mathcal{L}}(n) - \mu_{\mathcal{L}}^*(n))/3 + o(n)}.$$

Proof. Let \mathcal{F} denote the set of containers obtained by applying Lemma 7. Since every \mathcal{L} -free subset of $[n]$ lies in at least one of the $2^{o(n)}$ containers, it suffices to show that every $F \in \mathcal{F}$ houses at most $3^{(\mu_{\mathcal{L}}(n) - \mu_{\mathcal{L}}^*(n))/3 + o(n)}$ maximal \mathcal{L} -free subsets.

Let $F \in \mathcal{F}$. By Lemmas 7(i) and 8, $F = A \cup B$ where $|A| = o(n)$, $|B| \leq \mu_{\mathcal{L}}(n)$ and B is \mathcal{L} -free. Note that we can add all the elements of $\mathcal{M}_{\mathcal{L}}(n)$ to B (and thus F) whilst ensuring that $|B| \leq \mu_{\mathcal{L}}(n)$ and B is \mathcal{L} -free. So we may assume that $\mathcal{M}_{\mathcal{L}}(n) \subseteq B$.

Each maximal \mathcal{L} -free subset of $[n]$ in F can be found by picking a subset $S \subseteq A$ which is \mathcal{L} -free, and extending it in B . The number of ways of doing this is the number of ways of choosing the subset S multiplied by the number of ways of extending a fixed S in B , which we denote by $N(S, B)$. Since $|A| = o(n)$, there are $2^{o(n)}$ choices for S . It therefore suffices to show that for any $S \subseteq A$, we have $N(S, B) \leq 3^{(\mu_{\mathcal{L}}(n) - \mu_{\mathcal{L}}^*(n))/3}$.

Consider the link graph $G := L_S[B]$. Then by definition, $\mathcal{M}_{\mathcal{L}}(n)$ is an independent set in G . Thus, $\text{MIS}(G) = \text{MIS}(G \setminus \mathcal{M}_{\mathcal{L}}(n))$. Further, Lemma 12 and Theorem 13(i) imply that

$$N(S, B) \leq \text{MIS}(G) = \text{MIS}(G \setminus \mathcal{M}_{\mathcal{L}}(n)) \leq 3^{|B \setminus \mathcal{M}_{\mathcal{L}}(n)|/3} \leq 3^{(\mu_{\mathcal{L}}(n) - \mu_{\mathcal{L}}^*(n))/3},$$

as desired. \square

As mentioned in the introduction, Theorem 4 together with Theorem 9 shows that $f_{\max}(n, \mathcal{L})$ is significantly smaller than $f(n, \mathcal{L})$ for all homogeneous three-variable linear equations \mathcal{L} that are not translation-invariant. So in this sense it can be viewed as a generalisation of a result of Łuczak and Schoen [25] on sum-free sets.

Let \mathcal{L} denote the equation $px + y = z$ for some $p \in \mathbb{N}$. Notice that in this case we have $\mu_{\mathcal{L}}^*(n) = 0$ for $n > p$. The next result implies that if p is large then $f_{\max}(n, \mathcal{L})$ is close to the bound in Theorem 4. So for such equations \mathcal{L} , Theorem 4 is close to best possible.

Proposition 21. *Given $p, n \in \mathbb{N}$ where $p \geq 2$, let \mathcal{L} denote the equation $px + y = z$. Then*

$$f_{\max}(n, \mathcal{L}) \geq 3^{\mu_{\mathcal{L}}(n)/3 - 2pn/(3(p+1)(3p^2-1)) - p - 5}.$$

Proof. Given $p, n \in \mathbb{N}$, let \mathcal{L} denote the equation $px + y = z$. Set $s := \lfloor \frac{(p-1)n}{3p^2-1} \rfloor$ and $a := \lfloor \frac{n-s}{p} \rfloor$. Consider the link graph $G := L_{\{s, 2s\}}[a+1, a+3ps]$. Observe that:

$$\begin{aligned} 2s &\leq \frac{(2p-2)n}{3p^2-1} < \frac{n}{p+1} < \frac{(3p-1)n}{3p^2-1} = \frac{n}{p} - \frac{(p-1)n}{3p^3-p} \leq \frac{n-s}{p} < a+1; \\ a+3ps &= \left\lfloor \frac{n-s}{p} \right\rfloor + 3ps \leq \frac{n}{p} + \left(3p - \frac{1}{p}\right)s = \frac{n}{p} + \frac{3p^2-1}{p} \left\lfloor \frac{(p-1)n}{3p^2-1} \right\rfloor \leq \frac{n+n(p-1)}{p} = n. \end{aligned}$$

As a consequence, the sets $\{s, 2s\}$ and $[a+1, a+3ps]$ (a subset of $[\lfloor \frac{n}{p+1} \rfloor + 1, n]$) are disjoint \mathcal{L} -free sets in $[n]$, and so Lemma 15 implies that $f_{\max}(n, \mathcal{L}) \geq \text{MIS}(G)$. It remains to show that G contains at least $3^{\mu_{\mathcal{L}}(n)/3 - 2pn/(3(p+1)(3p^2-1)) - 6}$ maximal independent sets.

Observe that for each $i \in [ps]$ there is an edge in G between $a+i$ and $a+ps+i$ (since $\{s, a+i, a+i+ps\}$ is an \mathcal{L} -triple), an edge between $a+i+ps$ and $a+i+2ps$ (since $\{s, a+i+ps, a+i+2ps\}$ is an \mathcal{L} -triple) and an edge between $a+i$ and $a+i+2ps$ (since $\{2s, a+i, a+i+2ps\}$ is an \mathcal{L} -triple). Also since $a > (n-s)/p - 1$, we have $p(a+1) + s > n$ and hence there are no further edges in G .

Hence G is a collection of ps disjoint triangles, where 4 vertices in G have loops $((p+1)s, (p+2)s, (2p+1)s$ and $(2p+2)s)$. So G has at least 3^{ps-4} maximal independent sets. Now observe:

$$\begin{aligned} ps - 4 - \frac{\mu_{\mathcal{L}}(n)}{3} &= p \left\lfloor \frac{(p-1)n}{3p^2-1} \right\rfloor - 4 - \frac{n}{3} + \frac{1}{3} \left\lfloor \frac{n}{p+1} \right\rfloor \geq \left(\frac{p^2-p}{3p^2-1} - \frac{1}{3} + \frac{1}{3(p+1)} \right) n - p - 5 \\ &= \left(\frac{-2p}{3(p+1)(3p^2-1)} \right) n - p - 5, \end{aligned}$$

as required. \square

5.2. Upper bounds for $px + qy = z$. Let \mathcal{L} denote the equation $px + qy = z$ where $p \geq q$, $p \geq 2$ and $p, q \in \mathbb{N}$. For such \mathcal{L} , the next simple result provides an alternative bound to Theorem 4.

Lemma 22. *Let \mathcal{L} denote the equation $px + qy = z$ where $p \geq q$, $p \geq 2$ and $p, q \in \mathbb{N}$. Then $f_{\max}(n, \mathcal{L}) \leq f(\lfloor (n-p)/q \rfloor, \mathcal{L})$.*

Proof. Set $C := \lfloor \frac{n-p}{q} \rfloor$ and $B := \lfloor \frac{n-p}{q} \rfloor + 1, n$. In particular, B is \mathcal{L} -free. Notice that every maximal \mathcal{L} -free subset of $[n]$ can be found by selecting an \mathcal{L} -free subset $S \subseteq C$ and then extending it in B to a maximal one. Suppose we have such an \mathcal{L} -free subset S . By Lemma 12, the number of such extensions of S is at most $\text{MIS}(L_S[B])$.

For any \mathcal{L} -triple $\{x, y, z\}$ in $[n]$ satisfying $px + qy = z$, since $z \leq n$, we must have $x \leq \frac{n-q}{p}$ and $y \leq \frac{n-p}{q}$. Hence $x, y \in C$. This means that there are no \mathcal{L} -triples in $[n]$ which contain more than one element from B . Thus the link graph $L_S[B]$ must only contain isolated vertices and loops. So $L_S[B]$ has precisely one maximal independent set. Hence the number of maximal \mathcal{L} -free subsets of $[n]$ is bounded by the number of choices of S in C which are \mathcal{L} -free, i.e. $f(\lfloor (n-p)/q \rfloor, \mathcal{L})$. \square

Lemma 22 together with Theorems 3 and 9 immediately implies Theorem 6.

The next result gives a further upper bound on $f_{\max}(n, \mathcal{L})$ for certain linear equations \mathcal{L} . Notice that for such \mathcal{L} , Theorem 5 yields a better bound than Theorem 4.

Theorem 5. Let \mathcal{L} denote the equation $px + qy = z$ where $p \geq q \geq 2$ are integers so that $p \leq q^2 - q$ and $\gcd(p, q) = q$. Then

$$f_{\max}(n, \mathcal{L}) \leq 2^{(\mu_{\mathcal{L}}(n) - \mu_{\mathcal{L}}^*(n))/2 + o(n)}.$$

Proof. Let \mathcal{F} denote the set of containers obtained by applying Lemma 7. Since every \mathcal{L} -free subset of $[n]$ lies in at least one of the $2^{o(n)}$ containers, it suffices to show that every $F \in \mathcal{F}$ houses at most $2^{(\mu_{\mathcal{L}}(n) - \mu_{\mathcal{L}}^*(n))/2 + o(n)}$ \mathcal{L} -free sets.

Let $F \in \mathcal{F}$. By Lemmas 7(i) and 8, $F = A \cup B$ where $|A| = o(n)$, $|B| \leq \mu_{\mathcal{L}}(n)$ and B is \mathcal{L} -free. Note that we can add all the elements of $\mathcal{M}_{\mathcal{L}}(n)$ to B (and thus F) whilst ensuring that $|B| \leq \mu_{\mathcal{L}}(n)$ and B is \mathcal{L} -free. So we may assume that $\mathcal{M}_{\mathcal{L}}(n) \subseteq B$. By Theorem 1, $\min(B) = \lfloor \frac{n}{p+q} \rfloor - t$ for some non-negative integer $t < (\frac{p+q-1}{p+q+p/q}) \lfloor \frac{n}{p+q} \rfloor$ and $|B| \leq \lceil \frac{(p+q-1)n}{p+q} \rceil - \lfloor \frac{p}{q}t \rfloor$, or $|B| \leq \frac{(q^2+1)n}{q^2+q+1}$.

Case 1: $\min(B) = \lfloor \frac{n}{p+q} \rfloor - t$ for $0 \leq t < (\frac{p+q-1}{p+q+p/q}) \lfloor \frac{n}{p+q} \rfloor$. Write $F = X \cup Y$ where $Y \subseteq \lfloor \frac{n}{p+q} \rfloor + 1, n$ is \mathcal{L} -free, and $X \subseteq [1, \lfloor \frac{n}{p+q} \rfloor]$. Note that $|X| = t' + o(n)$ and $|Y| \leq \lceil \frac{(p+q-1)n}{p+q} \rceil - \lfloor \frac{p}{q}t \rfloor - t' + o(n)$ where $t' \leq t$. Also $\mathcal{M}_{\mathcal{L}}(n) \subseteq Y$. Choose $S \subseteq X$ to be \mathcal{L} -free. Consider the link graph $L_S[Y]$ and observe that by Lemma 12, $N(S, Y) \leq \text{MIS}(L_S[Y])$. (Recall $N(S, Y)$ denotes the number of extensions of S in Y to a maximal \mathcal{L} -free set.)

Since $p \leq q^2 - q$, by Lemma 14 $L_S[Y]$ is triangle-free. By definition, $\mathcal{M}_{\mathcal{L}}(n)$ is an independent set in $L_S[Y]$ and so $\text{MIS}(L_S[Y]) = \text{MIS}(L_S[Y \setminus \mathcal{M}_{\mathcal{L}}(n)])$. Therefore Theorem 13(ii) implies that $\text{MIS}(L_S[Y]) \leq 2^{(|Y| - |\mathcal{M}_{\mathcal{L}}(n)|)/2}$. Overall, this implies that the number of \mathcal{L} -free sets contained in F is at most

$$2^{|X|} \times 2^{(|Y| - |\mathcal{M}_{\mathcal{L}}(n)|)/2} \leq 2^{t' + o(n) + (\mu_{\mathcal{L}}(n) - \mu_{\mathcal{L}}^*(n) - \lfloor \frac{p}{q}t \rfloor - t')/2} \leq 2^{(\mu_{\mathcal{L}}(n) - \mu_{\mathcal{L}}^*(n))/2 + o(n)},$$

as desired.

Case 2: $|B| \leq \frac{(q^2+1)n}{q^2+q+1}$. In this case $|F| \leq \frac{(q^2+1)n}{q^2+q+1} + o(n)$. Choose any \mathcal{L} -free $S \subseteq A$ (note there are at most $2^{o(n)}$ choices for S). Consider the link graph $L_S[B]$ and observe by Lemma 12 that $N(S, B) \leq \text{MIS}(L_S[B])$. Similarly as in Case 1 we have that $\text{MIS}(L_S[B]) = \text{MIS}(L_S[B'])$ where $B' := B \setminus \mathcal{M}_{\mathcal{L}}(n)$. By Theorem 13(i),

$$\text{MIS}(L_S[B']) \leq 3^{|B'|/3} \leq 3^{((q^2+1)n/(3(q^2+q+1)) - \mu_{\mathcal{L}}^*(n))/3} \leq 2^{(\mu_{\mathcal{L}}(n) - \mu_{\mathcal{L}}^*(n))/2 + o(n)}.$$

The last inequality follows since $\mu_{\mathcal{L}}(n) = n - \lfloor n/(p+q) \rfloor$ and $\mathcal{M}_{\mathcal{L}}(n) = \{s : s > \lfloor (n-p)/q \rfloor, q \nmid s\}$ since $\gcd(p, q) = q$.

To see this, first note that

$$\mu_{\mathcal{L}}^*(n) = \frac{(q-1)^2 n}{q^2} - o(n).$$

Hence for the inequality to hold we require that

$$9^{((q^2+1)/(q^2+q+1)-(q^2-2q+1)/(q^2))} < 8^{((p+q-1)/(p+q)-(q^2-2q+1)/(q^2))}.$$

Let $a := \log_9 8$. This rearranges to give

$$p > \frac{(1-a)(q^4 - q) + q^3 + q^2}{(2a-1)q^3 + (a-1)(q^2 + q - 1)}.$$

Since $p \geq q$ it suffices to show that $(3a-2)q^3 + (a-2)(q^2 + q) + (2-2a) > 0$. This indeed holds since $q \geq 2$.

Overall, this implies that the number of \mathcal{L} -free sets contained in F is at most $2^{(\mu_{\mathcal{L}}(n) - \mu_{\mathcal{L}}^*(n))/2 + o(n)}$, as desired. \square

The proof of Theorem 5 actually generalises to some other equations $px + qy = z$ where $\gcd(p, q) \neq q$ (but still $p \leq q^2 - q$). However, in these cases Theorem 6 produces a better upper bound on $f_{\max}(n, \mathcal{L})$. The next result summarises when Theorem 4, 5 or 6 yields the best upper bound on $f_{\max}(n, \mathcal{L})$. We defer the proof to the appendix.

Proposition 23. *Let \mathcal{L} denote the equation $px + qy = z$ where $p \geq q$, $p \geq 2$ and $p, q \in \mathbb{N}$. Up to the error term in the exponent, the best upper bound on $f_{\max}(n, \mathcal{L})$ given by Theorems 4, 5 and 6 is:*

- (i) $f_{\max}(n, \mathcal{L}) \leq 3^{(\mu_{\mathcal{L}}(n) - \mu_{\mathcal{L}}^*(n))/3 + o(n)}$ if $\gcd(p, q) = q$, $p \geq q^2$, and either $q \leq 9$ or $10 \leq q \leq 17$ and $p < (a-1)(q^2 - q)/(q(2-a) - 1)$ where $a := \log_3(8)$;
- (ii) $f_{\max}(n, \mathcal{L}) \leq 2^{(\mu_{\mathcal{L}}(n) - \mu_{\mathcal{L}}^*(n))/2 + o(n)}$ if $\gcd(p, q) = q$ and $p \leq q^2 - q$;
- (iii) $f_{\max}(n, \mathcal{L}) \leq 2^{\mu_{\mathcal{L}}(\lfloor (n-p)/q \rfloor) + o(n)}$ otherwise.

5.3. Lower bounds for $px + qy = z$. The following result provides lower bounds on $f_{\max}(n, \mathcal{L})$ for all equations \mathcal{L} of the form $px + qy = z$ where $p \geq q \geq 2$.

Proposition 24. *Let \mathcal{L} denote the equation $px + qy = z$ where $p \geq q \geq 2$ are integers. Suppose that $n > 2p$. In each case $f_{\max}(n, \mathcal{L}) \geq 2^\ell$ where ℓ is defined as follows:*

- (i) $\ell := (n(q-1) - pq + q - 2q^2)/q^2$ if $p \geq q^2$,
- (ii) $\ell := (n(p-q) - p^2 + q^2 - 2pq)/(pq)$ if $q < p < q^2$,
- (iii) $\ell := (n - 6q)/2q$ if $p = q$.

Proof. For each case, we shall let $B := [\lfloor \frac{n}{p+q} \rfloor + 1, n]$, and consider the link graph $G := L_{\{1\}}[B]$. Since B and $\{1\}$ are \mathcal{L} -free, by Lemma 15 it suffices to show that there is an induced subgraph of G which contains at least 2^ℓ maximal independent sets. For each case we will find an induced perfect matching on 2ℓ vertices in G . (Note there are 2^ℓ maximal independent sets in such a matching.)

More specifically, for each case we shall find an interval $I := [a, b]$ for some $a, b \in V(G)$ and let $J := \{qi + p \mid i \in I\}$. Note that all edges in G (other than at most one loop) are of the form $\{i, qi + p\}$ and $\{i, pi + q\}$. By our choice of I and J , $G[I \cup J]$ will form a perfect matching on $2|I|$ vertices if the following conditions hold:

- (1) $qa + p > b$ (which ensures that $I \cap J = \emptyset$),
- (2) $qb + p \leq n$ (which ensures that $J \subseteq [n]$),
- (3) $pa + q > n$ (which ensures that the only edges in G are of the form $\{i, qi + p\}$),
- (4) $p + q < a$ (which ensures that there is no loop at a vertex in $G[I \cup J]$).

Notice that actually we do not require condition (3) to hold in the case when $p = q$. Indeed, this is because in this case an edge $\{i, pi + q\}$ in G is the same as the edge $\{i, qi + p\}$. Further, there is at most one loop in G (if $p + q \in B$). So even if (4) does not hold we will obtain an induced matching in G on $2|I| - 2$ vertices.

Thus, to obtain an induced matching in G on $2|I| - 2$ vertices it suffices to choose a and b so that (1)–(3) hold except when $p = q$ when we only require that (1) and (2) hold.

By choosing $b := \lfloor (n - p)/q \rfloor$, (2) holds since $qb + p = q\lfloor (n - p)/q \rfloor + p \leq q(n - p)/q + p = n$.

If $p \geq q^2$ then set $a := \lfloor (n - q)/q^2 \rfloor + 1$. Then $a \in B$ and further $pa + q \geq q^2a + q > q^2((n - q)/q^2) + q = n$ and $qa + p \geq qa + q^2 > q((n - q)/q^2) + q^2 = n/q - 1 + q^2 > \lfloor (n - p)/q \rfloor = b$. So (1) and (3) hold.

If $q < p < q^2$ then set $a := \lfloor (n - q)/p \rfloor + 1$. So $a \in B$. Further, $pa + q > p((n - q)/p) + q = n$ and $qa + p > q((n - q)/p) + p = qn/p - q^2/p + p > qn/q^2 - q + p > n/q > \lfloor (n - p)/q \rfloor = b$. So (1) and (3) hold.

If $p = q$ set $a := \lfloor n/(p + q) \rfloor + 1 = \lfloor n/(2q) \rfloor + 1 \in B$. Observe that $qa + q > qn/2q + q > n/2 > \lfloor (n - q)/q \rfloor = b$ since $q \geq 2$. So (1) holds.

Now calculating the size of the interval $I = [a, b]$ in each case proves the result:

- If $a = \lfloor (n - q)/q^2 \rfloor + 1$, then $|I| - 1 = \lfloor (n - p)/q \rfloor - (\lfloor (n - q)/q^2 \rfloor + 1) \geq (n - p)/q - 1 - (n - q)/q^2 - 1 = (n(q - 1) - pq + q - 2q^2)/q^2$.
- If $a = \lfloor (n - q)/p \rfloor + 1$, then $|I| - 1 = \lfloor (n - p)/q \rfloor - (\lfloor (n - q)/p \rfloor + 1) \geq (n - p)/q - 1 - (n - q)/p - 1 = (n(p - q) - p^2 + q^2 - 2pq)/(pq)$.
- If $a = \lfloor n/(p + q) \rfloor + 1$ then $|I| - 1 = \lfloor (n - p)/q \rfloor - (\lfloor n/(p + q) \rfloor + 1) \geq (n - p)/q - 1 - n/(p + q) - 1 = (pn - (p + 2q)(p + q))/(q(p + q)) = (qn - 6q^2)/(2q^2) = (n - 6q)/2q$.

□

Although the lower bounds in Proposition 24 do not meet the upper bounds in Theorems 5 and 6 in general, Theorem 5 and Proposition 24(iii) do immediately imply the following asymptotically exact result.

Theorem 25. *Let \mathcal{L} denote the equation $2x + 2y = z$. Then $f_{\max}(n, \mathcal{L}) = 2^{n/4 + o(n)}$.*

Since submitting this paper, we have also given a general upper bound on $f_{\max}(n, \mathcal{L})$ for equations \mathcal{L} of the form $px + qy = rz$ where $p \geq q \geq r$ are fixed positive integers (see [20]). In particular, our result shows that in the case when $p = q \geq 2$, $r = 1$ the lower bound in Proposition 24(iii) is correct up to an error term in the exponent.

6. CONCLUDING REMARKS

The results in the paper show that the parameter $f_{\max}(n, \mathcal{L})$ can exhibit very different behaviour depending on the linear equation \mathcal{L} . Indeed, Theorem 4 gives a ‘crude’ general upper bound on $f_{\max}(n, \mathcal{L})$ for all homogeneous three-variable linear equations \mathcal{L} . (It is crude in the sense that, in the proof, we do not use any structural information about the link graphs.) However, this bound is close to the correct value of $f_{\max}(n, \mathcal{L})$ for certain equations \mathcal{L} (Proposition 21). On the other hand, for many equations this bound is far from tight (Theorem 5). Further, for some equations ($x + y = z$ and $2x + 2y = z$) the value of $f_{\max}(n, \mathcal{L})$ is tied to the property that any triangle-free graph on n vertices contains at most $2^{n/2}$ maximal independent sets. Theorem 6 and upper bounds we have obtained since submitting this paper (see [20]) suggest though that the value of $f_{\max}(n, \mathcal{L})$ for other equations \mathcal{L} may depend on completely different factors. Further progress on understanding the possible behaviour of $f_{\max}(n, \mathcal{L})$ would be extremely interesting.

We conclude by briefly describing some results concerning equations with more than three variables. First observe the following simple proposition.

Proposition 26. Let \mathcal{L}_1 denote the equation $p_1x_1 + \cdots + p_kx_k = b$ where $p_1, \dots, p_k, b \in \mathbb{Z}$ and let \mathcal{L}_2 denote the equation $(p_1 + p_2)x_1 + p_3x_2 + \cdots + p_kx_{k-1} = b$. Then $\mu_{\mathcal{L}_1}(n) \leq \mu_{\mathcal{L}_2}(n)$.

The proposition is just a simple consequence of the observation that any solution to the equation \mathcal{L}_2 gives rise to a solution to the equation \mathcal{L}_1 . So all \mathcal{L}_1 -free subsets of $[n]$ are also \mathcal{L}_2 -free. Note that for the equations \mathcal{L} which satisfy the hypothesis of the following corollary, the interval $[\lfloor n/(p+q) \rfloor + 1, n]$ is \mathcal{L} -free. Hence by applying the above proposition along with Corollary 2, we attain the following result.

Corollary 27. Let \mathcal{L} denote the equation $a_1x_1 + \cdots + a_kx_k + b_1y_1 + \cdots + b_\ell y_\ell = c_1z_1 + \cdots + c_mz_m$ where the $a_i, b_i, c_i \in \mathbb{N}$ and $p' := \sum_i a_i$, $q' := \sum_i b_i$ and $r' := \sum_i c_i$. Let $t' := \gcd(p', q', r')$ and write $p := p'/t'$, $q := q'/t'$ and $r := r'/t'$. Suppose that $r = 1$. Then for sufficiently large n , we have $\mu_{\mathcal{L}}(n) = n - \lfloor n/(p+q) \rfloor$.

One can define a link hypergraph $L_S[B]$ analogous to the notion of a link graph defined in Section 2.3 (i.e. now hyperedges correspond to solutions to \mathcal{L} involving at least one element of S). We remark that the removal and container lemmas of Green [18] that we applied do hold for homogeneous linear equations on more than three variables. By arguing as in Lemma 22 (but by considering a link hypergraph), one can obtain the following simple result.

Proposition 28. Let \mathcal{L} denote the equation $p_1x_1 + \cdots + p_sx_s = rz$ where $p_1 \geq p_2 \geq \cdots \geq p_s > r \geq 1$ are positive integers. Then $f_{\max}(n, \mathcal{L}) \leq f(\lfloor rn/p_s \rfloor, \mathcal{L})$.

In [20] we obtain further results concerning the number of maximal solution-free sets for linear equations with more than three variables. However the proof method does not use structural results such as Theorem 13, and only work for *some* linear equations. Obtaining similar structural results for the number of maximal independent sets in (non-uniform) hypergraphs would help to attain (general) upper bounds for the number of maximal solution-free sets.

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APPENDIX A.

In this appendix we give the proof of Claim 18 and Proposition 23.

A.1. Proof of Claim 18. We use induction on k . Recall that $p \geq q \geq 2$. For the base case $k = 6$ we directly calculate (4). First note that

$$\frac{q^2 + 1}{q^2 + q + 1} - \frac{q^7 - q^6 + q^5 - q^4 + q^3 - q^2 + q - 1 + p(q^6 + q^4 + q^2 + 1)}{q^7 + p(q^6 + q^5 + q^4 + q^3 + q^2 + q + 1)}$$

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$$= \frac{(q^6 + (p-1)q^5 + q^4 + (p-1)q^3 + q^2 + 1)}{(q^2 + q + 1)(q^7 + p(q^6 + q^5 + q^4 + q^3 + q^2 + q + 1))},$$

and so we have

$$\begin{aligned} & \left(\frac{q^2 + 1}{q^2 + q + 1} - C_6 \right) (q^6 + p(q^5 + q^4 + q^3 + q^2 + q + 1)) \\ &= \frac{(q^6 + (p-1)q^5 + q^4 + (p-1)q^3 + q^2 + 1)(q^6 + p(q^5 + q^4 + q^3 + q^2 + q + 1))}{(q^2 + q + 1)(q^7 + p(q^6 + q^5 + q^4 + q^3 + q^2 + q + 1))}. \end{aligned}$$

Since $p \geq q \geq 2$ every power of q in the numerator has a coefficient of at least 1 in both expressions, hence the numerator as a single polynomial in q has positive coefficients. Hence we can make our fraction smaller by dropping lower powers of q . We then make further use of $p \geq q \geq 2$ to get the desired result:

$$\begin{aligned} & \frac{(q^6 + (p-1)q^5 + q^4 + (p-1)q^3 + q^2 + 1)(q^6 + p(q^5 + q^4 + q^3 + q^2 + q + 1))}{(q^2 + q + 1)(q^7 + p(q^6 + q^5 + q^4 + q^3 + q^2 + q + 1))} \\ & \geq \frac{q^{12} + (2p-1)q^{11} + (p^2+1)q^{10} + (p^2+2p-1)q^9}{(q^2 + q + 1)(q^7 + p(q^6 + q^5 + q^4 + q^3 + q^2 + q + 1))} \\ & \geq \frac{q^{12} + (2p-1)q^{11} + (p^2+1)q^{10} + (p^2+2p-1)q^9}{(p+1)q^{10}} \\ & = \frac{q^2 + (2p-1)q + (p^2+1)}{p+1} + \frac{p^2+2p-1}{(p+1)q} \geq \frac{p^2+4p+3}{p+1} + \frac{p^2+p}{(p+1)q} = p+3+p/q \geq 6 = k. \end{aligned}$$

For the inductive step, assume that (4) holds for k . It suffices to show that $C_k \geq C_{k+1}$ as then the result holds for $k+1$:

$$\begin{aligned} & \left(\frac{q^2 + 1}{q^2 + q + 1} - C_{k+1} \right) \left(q^{k+1} + p \sum_{i=0}^k q^i \right) \geq \left(\frac{q^2 + 1}{q^2 + q + 1} - C_k \right) \left(q^{k+1} + p \sum_{i=0}^k q^i \right) \\ & \geq q \left(\frac{q^2 + 1}{q^2 + q + 1} - C_k \right) \left(q^k + p \sum_{i=0}^{k-1} q^i \right) \geq qk \geq k+1. \end{aligned}$$

For k even, we have $C_k = C_{k+1}$ by definition. For k odd, consider the following calculations:

- (i) $D_1 := q^{k+2} \left(\sum_{i=0}^k (-1)(-q)^i \right) - q^k \left(\sum_{i=0}^{k+2} (-1)(-q)^i \right) = -q^{k+1} + q^k,$
- (ii) $D_2 := pq^{k+2} \left(\sum_{i=0}^{(k-1)/2} q^{2i} \right) - pq^k \left(\sum_{i=0}^{(k+1)/2} q^{2i} \right) = -pq^k,$
- (iii) $D_3 := p \left(\sum_{i=0}^{k+1} q^i \right) \left(\sum_{i=0}^k (-1)(-q)^i \right) - p \left(\sum_{i=0}^{k-1} q^i \right) \left(\sum_{i=0}^{k+2} (-1)(-q)^i \right) = pq^{k+1} - pq^k,$
- (iv) $D_4 := p^2 \left(\sum_{i=0}^{k+1} q^i \right) \left(\sum_{i=0}^{(k-1)/2} q^{2i} \right) - p^2 \left(\sum_{i=0}^{k-1} q^i \right) \left(\sum_{i=0}^{(k+1)/2} q^{2i} \right) = p^2 q^k.$

Using these we have

$$\begin{aligned}
C_k - C_{k+1} &= \frac{\left(\sum_{i=0}^k (-1)(-q)^i\right) + p\left(\sum_{i=0}^{(k-1)/2} q^{2i}\right)}{q^k + p\left(\sum_{i=0}^{k-1} q^i\right)} - \frac{\left(\sum_{i=0}^{k+2} (-1)(-q)^i\right) + p\left(\sum_{i=0}^{(k+1)/2} q^{2i}\right)}{q^{k+2} + p\left(\sum_{i=0}^{k+1} q^i\right)} \\
&= \frac{D_1 + D_2 + D_3 + D_4}{\left(q^k + p\left(\sum_{i=0}^{k-1} q^i\right)\right)\left(q^{k+2} + p\left(\sum_{i=0}^{k+1} q^i\right)\right)} \\
&= \frac{(p-1)q^{k+1} + (p^2 - 2p + 1)q^k}{\left(q^k + p\left(\sum_{i=0}^{k-1} q^i\right)\right)\left(q^{k+2} + p\left(\sum_{i=0}^{k+1} q^i\right)\right)} \geq 0,
\end{aligned}$$

where the last inequality follows since $p, q \geq 2$. \square

A.2. Proof of Proposition 23. Suppose that $\gcd(p, q) = q$. To prove (ii) it suffices to show that

$$\mu_{\mathcal{L}}(n) - \mu_{\mathcal{L}}^*(n) \leq 2\mu_{\mathcal{L}}(\lfloor (n-p)/q \rfloor) + o(n).$$

Since $\mu_{\mathcal{L}}(n) = (p+q-1)n/(p+q) + o(n)$, $\mu_{\mathcal{L}}(\lfloor (n-p)/q \rfloor) = (p+q-1)n/q(p+q) + o(n)$ and $\mu_{\mathcal{L}}^*(n) = (q-1)^2n/q^2 + o(n)$, it is easy to check that this inequality holds.

To prove (iii) in the case where $t := \gcd(p, q) \neq q$, it certainly suffices to show that $2\mu_{\mathcal{L}}(\lfloor (n-p)/q \rfloor) \leq \mu_{\mathcal{L}}(n) - \mu_{\mathcal{L}}^*(n) + o(n)$. In this case we have $\mu_{\mathcal{L}}^*(n) = (q-1)(t-1)/(qt) + o(n)$, and hence it suffices to show that $t \leq (pq + q^2 - p - q)/(p + 2q - 2)$. First note that $t \leq q/2$ and so $q \neq 1$. Now observe that $t(p + 2q - 2) \leq q(p + 2q - 2)/2 = pq/2 + q^2 - q \leq pq + q^2 - p - q$ and so our inequality on t holds as required.

To prove (iii) in the case where $\gcd(p, q) = q$ and $p \geq q^2$, it suffices to show that

$$2 \frac{(p+q-1)n}{(p+q)q} \leq 3 \frac{(p+q-1)n}{3(p+q)} - \frac{(q-1)^2n}{3q^2}.$$

Let $a := \log_3(8)$. The inequality can be rearranged to give

$$p((2-a)q - 1) \geq (a-1)(q^2 - q).$$

If $q \geq 10$ then $((2-a)q - 1)$ is positive and so we require $p \geq (a-1)(q^2 - q)/((2-a)q - 1)$. Note that for $q \geq 18$ this always holds since $p \geq q^2 \geq (a-1)(q^2 - q)/((2-a)q - 1)$.

To prove (i), suppose that $\gcd(p, q) = q$. It suffices to show that

$$3 \frac{(p+q-1)n}{3(p+q)} - \frac{(q-1)^2n}{3q^2} \leq 2 \frac{(p+q-1)n}{(p+q)q},$$

or rearranging

$$p((2-a)q - 1) \leq (a-1)(q^2 - q).$$

If $q \leq 9$ then $((2-a)q - 1)$ is negative and so the inequality holds as the right hand side is non-negative. If $10 \leq q \leq 17$ then the inequality holds if $p \leq (a-1)(q^2 - q)/((2-a)q - 1)$. \square